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LJUSTERNIK-SCHNIRELMAN THEORY ON GENERAL LEVEL SETS.(U)
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LJUSTERNIK-SCHNIRELMAN THEORY ON GENERAL LEVEL SETS

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LJUSTERNIK-SCHNIRELMAN THEORY ON GENERAL LEVEL SETS

10 Eberhard Zeidler[†]

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ABSTRACT

Let a, b functionals on a real Banach space X. We consider the nonlinear eigenvalue problem

$$a'(u) = \lambda b'(u), \quad \lambda \in \mathbb{R}, \; u \in N_\alpha \equiv \{u \in X : b(u) = \alpha\}$$

for fixed α. We allow that a', b' are indefinite and the level set N_α is unbounded.

We get finite and infinite lower bounds for the number of eigenvectors on N_α depending on α. Furthermore, we study the weak convergence of the eigenvectors u against zero and the convergence of the eigenvalues λ against zero.

Our applications are concerned with eigenvalue problems for nonlinear elliptic partial differential equations where the principal elliptical part is indefinite, and with Hammerstein integral equations where the kernel has eigenvalues of different signs.

The main abstract theorems in Section 5 provide a general formulation of the Ljusternik-Schnirelman theory in Banach spaces in the constrained case.

AMS (MOS) Subject Classification: 47H15

Key Words: Nonlinear operator, Eigenvalues, Elliptic partial differential equations, Hammerstein equations.

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SIGNIFICANCE AND EXPLANATION

The main task of Ljusternik-Schnirelman Theory is to find critical points of functionals which are not merely maxima or minima, and to give lower bounds for the number of such critical points. The theory enables results to be obtained in a variety of physical problems involving nonlinear equations which cannot be handled by simpler traditional methods.

In this paper we obtain some new estimates of such lower bounds for eigenvalue problems for nonlinear elliptic partial differential equations, where the principal part is indefinite. Earlier results of Krasnosel'skii are not applicable to partial differential equations.

One new feature of this study is that saddle points can be dealt with as well as extreme values. Existing results apply mostly to cases where the level sets (see abstract) are sphere-like. The cases covered in the present paper include hyperboloid-like level sets as well.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

LJUSTERNIK-SCHNIRELMAN THEORY ON GENERAL LEVEL SETS

Eberhard Zeidler[+]

Introduction.

In order to explain the basic ideas of this paper let us consider the nonlinear boundary value problem

$$(P^*)\qquad \begin{aligned} -\Delta u + mu &= \mu f'(u) \quad \text{on } G \\ u &= 0 \quad \text{on } \partial G \end{aligned}$$

where G is a bounded nonempty domain in $\mathbb{R}^n$. Suppose $f \in C^1(\mathbb{R})$, and f' is odd. m is a given real number.

We are looking for eigensolutions (u,μ).

Define

$$a(u) = \int_G f(u(x))dx,\ 2b(u) = \int_G (\nabla u)^2 + mu^2 dx .$$

If we set $X = \overset{0}{W}{}_2^1(G)$, then under growth conditions of f, f', the generalized problem belonging to (P^*) reads as

$$(P)\qquad \mu a'(u) = b'(u),\ u \in N_\alpha,\ \mu \in \mathbb{R}$$

where $N_\alpha = \{u \in X : b(u)\} = \alpha$. The condition $u \in N_\alpha$ is a normalization condition for the eigenvector u.

The following observation is crucial. Let μ_1 be the smallest eigenvalue of (P^*) in the special linear case $m = 0$, $f'(u) \equiv u$. Now consider the nonlinear problem (P^*). Then, the following holds:

(i) If $m + \mu_1 > 0$ (e.g. $m \geq 0$) then N_α with $\alpha > 0$ is a spherelike set.

(ii) If $m + \mu_1 < 0$, then it is meaningful to consider N_α for $\alpha > 0$ and $\alpha < 0$. In both cases N_α is not spherelike because there exist subspaces X_1, X_2 of X with $b(u) > 0$ on X_1 and $b(u) < 0$ on X_2. Roughly speaking, N_α has the structure of a hyperboloid (see Section 8).

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The goal of this paper is to formulate the Ljusternik-Schnirelman theory for (P) in such a way that we can consider boundary value problems as well of the type (i) as (ii). Our applications to nonlinear boundary value problems and some of our abstract results seem to be new or generalize previous results of other authors. In a forthcoming paper we will consider applications in elasticity.

If we consider (P) in terms of definitness, then we have the following cases:

(I) a', b' are definite

(II) a' is indefinite, b' is definite

(III) a', b' are indefinite.

Most papers in Ljusternik-Schnirelman theory are concerned with (I).

Amann [2] was the first who considered (II). A crucial role in his approach plays a local version of the global Palais-Smale condition (PS) (see [23]) which goes back to Krasnosel'skii [21]. Stronger results in case (II) were obtained in Zeidler [34] - [36]. The results in [35], [36] are maximal compared with the corresponding linear problem $\mu a'(u) = u$.

This paper deals with the case (III). We generalize earlier results due to Krasnosel'skii [20], [21] Chapter VI, 16 and Borisovič [4] which are not applicable to boundary value problems.

In order to explain the differences between the cases (I), (II), (III) let us consider the linear problem

$$(P_L)\qquad \mu Au = Bu,\ u \in N_\alpha,\ \mu \in \mathbb{R}$$

corresponding to (P) where X is a real Hilbert space with the scalar product $(\ldots)$, $A : X \to X$ is a selfadjoint completely continuous operator, $B = I - 2Q$, $Q : X \to X$

is a linear orthogonal projector on a finite-dimensional subspace. If we set $2a(u) = (Au,u)$, $2b(u) = (u - 2Qu,u)$, then (P_L) is a special case of (P) (I = identity).

Now:

(i) case (I) corresponds to $(Au,u) > 0$ if $u \neq 0$, and $Q = 0$

(ii) case (II) corresponds to $Q = 0$

(iii) case (III) corresponds to $Q \neq 0$.

In case (I), (II), N_α is a ball if $\alpha > 0$.

In case (III) every $u \in X$ allows the decomposition $u = u_1 + u_2$, $u_1 \in Q(X)$, $u_2 \in (I - Q)(X)$. Hence

$$N_\alpha = \{u \in X : \|u_2\|^2 - \|u_1\|^2 = 2\alpha\} .$$

Such sets will be called hyperboloids if $\alpha \neq 0$.

This paper has been influenced by the papers of Krasnosel'skii [21], Schwartz [29], Palais [24], Coffman [11], Clark [10], Browder [9], Fučik, Nečas [16], Rabinowitz [27], Ambrosetti, Rabinowitz [3], Fadell, Rabinowitz [14], [15], Dancer [12], Zeidler [35], [36].

Our paper is organized as follows:

Notation

1. The local Palais-Smale condition
2. Deformations on the level set
3. A critical point principle
4. The index
5. Main theorems
6. Proofs of the main theorems
7. An important special case

8. Applications to hyperboloids
9. Applications to Hammerstein equations
10. Applications to Hammerstein integral equations
11. Applications to elliptic differential equations.

The essential feature of our main results can be seen in Section 7 (Proposition 6a, 6b).

We will come back to the boundary value problem (P*) in Section 11.

Basic ideas of this paper are:

a) the construction of a deformation on N_α via generalized pseudogradient vector fields (see Proposition 1 in Section 2)

b) the proof of $\beta_m^{\pm} \to 0$ as $m \to \infty$ ($\beta_m^{\pm}$ are the critical levels; see Theorem 2 in Section 5)

c) the index argument in the proof of Proposition 6b in Section 7 showing that the critical levels are finite beginning with a certain number.

The proof in a) is a modification of proofs in Clark [10] and Rabinowitz [26]. The idea of pseudogradient vector fields is due to Palais [24]. The proof in b) is contained in Zeidler [16] and employs a recent result due to Dancer [12] on the existence of nonlinear projectors in Banach spaces combined with ideas in Fučik, Nečas [16], [17].

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Notation

Our notation follows Zeidler [36].

The dual space of a Banach space X will be denoted by X^*. (u^*,v) means $u^*(v)$ if $u^* \in X^*$, $v \in X$. Strong convergence (resp. weak convergence) in X will be denoted by $\to$ (resp. $\rightharpoonup$).

$\mathbb{R}$ (resp. $\mathbb{N}$) denotes the set of all real numbers (resp. positive integers).

Let $a : X \to \mathbb{R}$ be a functional on X. $a \in C^k(X,\mathbb{R})$ means that $a(\cdot)$ is k times continuously differentiable (in the sense of Fréchet). If the Gateaux-derivative $a' : X \to X^*$ is continuous on X, then $a \in C^1(X,\mathbb{R})$ (see e.g. [32] p. 44).

Suppose $A : X \to X^*$ is an operator from X into X^* and X is reflexive. A is said to be completely continuous iff A maps bounded sets into relatively compact sets. A is said to be strongly continuous if $u_n \rightharpoonup u$ as $n \to \infty$ in X implies $u_n \to u$ in X^*. A is said to be bounded iff A maps bounded sets into bounded sets.

We have (see e.g. [33], p. 121):

A strongly continuous $\Rightarrow$ A completely continuous

$\Rightarrow$ A bounded.

A completely continuous, linear

$\Rightarrow$ A strongly continuous.

A is said to be uniformly monotone iff there exists a strictly increasing continuous function $\psi : [0,\infty[\to [0,\infty[$ with $\psi(0) = 0$, $\psi(t) \to +\infty$ as $t \to +\infty$ and

(1) $$(Au - Av, u - v) \geq \psi(\|u - v\|)\|u - v\|$$

for all $u, v \in X$.

A satisfies condition $(S)_1$ iff

$$u_n \rightharpoonup u,\ Au_n \to v \Rightarrow u_n \to u\ (n \to \infty) .$$

A satisfies $(S)_+$ iff

$$u_n \rightharpoonup u,\ \overline{\lim_{n\to\infty}}\,(Au_n - Au, u_n - u) \leq 0$$

$$\Rightarrow u_n \to u\ (n \to \infty) .$$

We have: $(S)_+ \Rightarrow (S)_1$.

Furthermore,

A uniformly monotone, A_1 completely continuous

$\Rightarrow A + A_1$ satisfies $(S)_+$, $(S)_1$ (see [33], p. 121).

A is said to be odd (resp. even) iff $A(-u) = -Au$ (resp. $A(-u) = Au$) for all $u \in X$.

$a : X \to \mathbb{R}$ is said to be weakly continuous iff $u_n \rightharpoonup u$ implies $a(u_n) \to a(u)$ $(n \to \infty)$.

If $a \in C^1(X,\mathbb{R})$, then $a' : X \to X^*$. If a is even, then a' is odd. If a' is strongly continuous, then a is weakly continuous (see e.g. [34], p. 66).

The identity operator in X is denoted by I.

The proofs for all the results stated above and further information about properties of nonlinear operators may be found e.g. in Zeidler [32] - [34].

1. The local Palais-Smale condition.

In this Section we make the following Assumption I_α.

(2) α is a fixed real number.

(3) X is a real Banach space.

(4) $a, b \in C^1(X,\mathbb{R})$ are given functionals.

(5) $\inf_{u \in K} |(b'(u),u)| > 0$ on every bounded subset K of

$N_\alpha := \{u \in X : b(u) = \alpha\}$.

(6) $b' : X \to X^*$ is bounded and local Lipschitz continuous on N_α.

(7) $a^{-1}(M) \cap N_\alpha$ is bounded if M is bounded in $\mathbb{R}$.

Under these assumptions, N_α is a Banach manifold over X. The tangent space T_u in an arbitrary point $u \in N_\alpha$ is given by $T_u = \{h \in X : (b'(u),h) = 0\}$ (see e.g. Zeidler [33], Theorem 41.1,4)). Now define, for all $u \in N_\alpha$,

$\bar{a}'(u) := a'(u) - \lambda(u)b'(u)$, $\lambda(u) = \dfrac{(a'(u),u)}{(b'(u),u)}$.

Lemma 1. Let $u \in N_\alpha$. The following three conditions are all equivalent:

(i) $(a'(u),h) = 0$ for all $h \in T_u$.

(ii) $\bar{a}'(u) = 0$.

(iii) There exists $\lambda \in \mathbb{R}$ with $a'(u) - \lambda b'(u) = 0$.

If (i), then u is said to be a critical point of $a(\cdot)$ on N_α.

Proof. Define, for all $u \in N_\alpha$, $v \in X$,

(8) $$P_u v = v - \frac{(b'(u),v)}{(b'(u),u)} u \quad .$$

P_u is a linear continuous projection from X onto the tangent space T_u.

Now, Lemma 1 follows from

(9) $$(\bar{a}'(u), (I - P_u)v) = 0, \ (\bar{a}'(u), P_u v) = (a'(u), P_u v) \quad .$$

q.e.d.

Definition 1. Let β be a fixed real number. The functional $a(\cdot)$ is said to satisfy a local Palais-Smale condition (PS_β) on N_α iff

(PS_β) every sequence (u_n) on N_α with $a(u_n) \to \beta$, $\bar{a}'(u_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

$a(\cdot)$ is said to satisfy (PS) (resp. (PS^+), (PS^-)) on N_α iff $a(\cdot)$ satisfies (PS_β) for all $\beta \in \mathbb{R}$ (resp. all $\beta > 0$, $\beta < 0$).

The following Lemma supplies a useful sufficient condition for (PS_β).

Lemma 2. (PS_β) holds under the following conditions:

(i) Assumption I_α is satisfied, X is reflexive.

(ii) b' satisfies $(S)_1$.

(iii) a' is strongly continuous on X.

(iv) If $u_n \rightharpoonup u$ as $n \to \infty$ and (u_n) on N_α with $a(u) = \beta$, then $a'(u) \neq 0$.

Corollary 1. Suppose $a, b \in C^1(X,\mathbb{R})$, X is a finite-dimensional real Banach space, and (7). Then (PS) holds.

Proof. Let (u_n) be a sequence on N_α with $a(u_n) \to \beta$, $\bar{a}'(u_n) \to 0$ as $n \to \infty$. (u_n) is bounded by (7). We choose a subsequence $u_{n'} \rightharpoonup u$. $a(\cdot)$ is weakly continuous by (iii). Hence $a(u) = \beta$ and $a'(u) \neq 0$ by (iv).

The sequence

$$\lambda(u_{n'}) = (a'(u_{n'}), u_{n'})/(b'(u_{n'}), u_{n'})$$

is bounded by (5), (iii). Without any loss of generality we can assume $\lambda(u_{n'}) \to \lambda_0$ as $n \to \infty$. Now $\bar{a}'(u_{n'}) = a'(u_{n'}) - \lambda(u_{n'})b'(u_{n'}) \to 0$, $a'(u_{n'}) \to a'(u)$ and $a'(u) \neq 0$ yield $\lambda_0 \neq 0$. Hence

$$u_{n'} \rightharpoonup u, \ b'(u_{n'}) \to \lambda_0^{-1} a'(u) \quad \text{as} \quad n \to \infty \quad ,$$

i.e. $u_{n'} \to u$ by (ii).

q.e.d.

The proof of Corollary 1 is simpler since in finite-dimensional Banach spaces weak convergence and strong convergence coincide.

2. Deformations on the level set N_α.

A deformation d on N_α is a continuous map $d : N_\alpha \times [0,1] \to N_\alpha$ such that $d(u,0) = u$ for all $u \in N_\alpha$.

The following Proposition will play an important role. We set

$$E_\beta = \{u \in N_\alpha : \bar{a}'(u) = 0,\ a(u) = \beta\},\ a_*^{-1}(M) = a^{-1}(M) \cap N_\alpha \ .$$

Proposition 1. Suppose:

(i) Assumption I_α is satisfied.

(ii) $a(\cdot)$ satisfies (PS_β) for a fixed $\beta \in \mathbb{R}$.

Then: For any given open set $U \supset E_\beta$, there exists a deformation d of N_α and a real number $\varepsilon > 0$ such that

(10) $$d(u,1) = u \quad \text{if} \quad u \notin a_*^{-1}([\beta - 2\varepsilon, \beta + 2\varepsilon])$$

(11) $u \to d(u,t)$ is a homeomorphism of N_α onto N_α for all $t \in [0,1]$.

(12) $$a(d(u,t)) \geq a(d(u)) \quad \text{on} \quad N_\alpha \times [0,1] \ .$$

(13) $$a(u) \geq \beta - \varepsilon,\ u \in N_\alpha - U \quad \text{implies} \quad a(d(u,1)) \geq \beta + \varepsilon \ .$$

(14) If a, b are even, then d is odd in u.

Corollary 2. Proposition 1 remains valid if we replace "$\geq$" by "$\leq$" and ε by $-\varepsilon$.

The proof is a slight modification of the proof of Theorem 1.9 in Rabinowitz [26] (see also Clark [10]) and employs a modification of the idea of a pseudo-gradient vector field (see Palais [24], Browder [9]).

Proof. Let U_δ be an open δ-neighborhood of E_β which is symmetric if a, b are even. Since (PS_β), E_β is compact. Choose a small $\delta > 0$ such that $E_\beta \subset U_\delta \subset U$. It suffices to prove Proposition 1 in the case $U = U_\delta$. If $E_\beta = \phi$, $U = \phi$, then set $U_\delta = \phi$.

Define

$$\|\bar{a}'(u)\|_* = \sup\{|(\bar{a}'(u),h)| : h \in T_u,\ \|h\| = 1\} \ .$$

Lemma 3. For all $u \in N_\alpha$,

(15) $$\|\bar{a}'(u)\|_* \leq \|\bar{a}'(u)\| \leq \left(1 + \frac{\|b'(u)\|\ \|u\|}{|(b'(u),u)|}\right) \|\bar{a}'(u)\|_* ,$$

where $\|\bar{a}'(u)\|$ denotes the usual operator norm.

Proof. The first inequality is obvious. The second inequality follows from

$$(\bar{a}'(u),h) = (\bar{a}'(u),P_u h) \qquad \text{(see (9))}$$

$$|(\bar{a}'(u),h)| \leq \|\bar{a}'(u)\|_*\ \|P_u h\|$$

$$\|P_u h\| \leq \|h\| \left(1 + \frac{\|b'(u)\|\ \|u\|}{|(b'(u),u)|}\right) \qquad \text{(see (8))}$$

for all $u \in N_\alpha$, $h \in X$.

q.e.d.

Lemma 4. There exists a constant $k > 0$ such that

(16) $$\|\bar{a}'(u)\|_* \geq k \|\bar{a}'(u)\| \quad \text{on} \quad a_*^{-1}([\beta - 1, \beta + 1]) \ .$$

Proof. See (15), (5), (6), (7).

q.e.d.

Lemma 5. If A, B are closed nonempty subsets of X with $A \cap B = \phi$, then there exists a local Lipschitz continuous functional $g : X \to \mathbb{R}$ with

$$g(u) = 0 \ \text{on} \ A,\ g(u) = 1 \ \text{on} \ B,\ 0 \leq g(u) \leq 1 \ \text{on} \ X \ .$$

If A, B are symmetric, then g is even.

Proof. Choose

$$g(u) = \operatorname{dist}(u,A)/(\operatorname{dist}(u,A) + \operatorname{dist}(u,B)) \ .$$

q.e.d.

Lemma 6. There exist constants $\tau, \varepsilon > 0$ such that

(17) $$\|\bar{a}'(u)\|_* \geq \tau > 0 \quad \text{on} \quad a_*^{-1}([\beta - 2\epsilon, \beta + 2\epsilon]) - U_{\delta/8}$$

(18) $$0 < \epsilon < \min\left(\frac{1}{8}, \frac{\tau^2 k^2}{8}, \frac{k^2}{32}, \frac{\delta k^2 \tau}{96}\right) .$$

Proof. Since we can pass to a smaller ϵ, it suffices to prove (17).

If (17) is not true, then there exists a sequence (u_n) on $N_\alpha - U_{\delta/8}$ with

$$a(u_n) \to \beta, \ \|\bar{a}'(u_n)\|_* \to 0 \quad \text{as} \quad n \to \infty .$$

Hence $\|\bar{a}'(u_n)\| \to 0$ by (16). (PS_β) shows $u_{n'} \to u$, i.e. $u \in E_\beta$, $u \notin U_{\delta/8}$. This contradicts $E_\beta \subset U_{\delta/8}$.

q.e.d.

Lemma 7. There exists a modified pseudogradient vector field $p : N_\alpha \to X$, i.e. for all $u \in N_\alpha$,

(19) $$\|p(u)\| \leq 2\|\bar{a}'(u)\|, \ p(u) \in T_u$$

(20) p is local Lipschitz continuous on $\{u \in N_\alpha : \bar{a}'(u) \neq 0\}$.

(21) $$\langle \bar{a}'(u), p(u) \rangle \geq 0 \quad \text{on} \quad N_\alpha$$

(22) $$\langle \bar{a}'(u), p(u) \rangle \geq \frac{1}{2} k^2 \|\bar{a}'(u)\|^2 \quad \text{on} \quad a_*^{-1}([\beta - \epsilon, \beta + \epsilon]) - U_{\delta/8}$$

(23) p is odd if a, b are even.

Proof. Set $V = a_*^{-1}([\beta - \epsilon, \beta + \epsilon]) - U_{\delta/8}$. V is closed and bounded by (7).

If $\bar{a}'(u) = 0$ we define $p(u) = 0$.

If $\bar{a}'(u_i) \neq 0$, $u_i \notin V$, we choose a neighborhood U_i of u_i with $U_i \cap V = \emptyset$ and set $z_i(u) = 0$ on U_i.

If $\bar{a}'(u_i) \neq 0$ and $u_i \in V$, then $\|a'(u_i)\|_* \neq 0$ by (9). Therefore, we can choose $h_i \in T_{u_i}$ such that

$$\|h_i\| = \|\bar{a}'(u_i)\|_*, \ \langle \bar{a}'(u_i), h_i \rangle > \frac{1}{2} \|\bar{a}'(u_i)\|_*^2 .$$

By Lemma 3, 4, (18),

$$\|h_i\| = \|\bar{a}'(u_i)\|_* \leq \|\bar{a}'(u_i)\|$$

$$\langle \bar{a}'(u_i), h_i \rangle > \frac{1}{2} k^2 \|\bar{a}'(u_i)\|^2 .$$

Define $z_i(u) = P_u h_i$. Obviously, $z_i(u_i) = h_i$. Then, there exists an open neighborhood U_i of u_i such that, for all $u \in U_i$,

$$\|z_i(u)\| \leq 2\|\bar{a}'(u)\|$$

$$\langle \bar{a}'(u), z_i(u) \rangle \geq \frac{1}{2} k^2 \|a'(u)\|^2 .$$

Since every subset of a Banach space is paracompact, we can assume that U_i is a locally finite covering of $N_\alpha - \{u \in N_\alpha : \bar{a}'(u) \neq 0\}$.

Define $\rho_i(u) = \mathrm{dist}(u, X - U_i)$ and

$$\eta_i(u) = \frac{\rho_i(u)}{\sum_i \rho_i(u)} .$$

Then ρ_i is Lipschitz continuous and η_i is locally Lipschitz continuous. Observe

$$\eta_i(u) \geq 0 \quad \text{on} \quad X \quad \text{and} \quad \sum_i \eta_i(u) = 1 .$$

Now, for all $u \in N_\alpha$ with $\bar{a}'(u) \neq 0$ we set

$$p(u) = \sum_i \eta_i(u) z_i(u) .$$

Then, p has all properties claimed in Lemma 7.

If a, b are even, we replace $p(u)$ by $\frac{1}{2}(p(u) - p(-u))$.

q.e.d.

Lemma 8. There exists a local Lipschitz continuous vector field $v : N_\alpha \to X$ such that

(24) $$0 \leq \|v(u)\| \leq 1 \quad \text{and} \quad v(u) \in T_u \quad \text{on} \quad N_\alpha$$

(25) $$v(u) = 0 \quad \text{if} \quad u \notin a_*^{-1}([\beta - 2\epsilon, \beta + 2\epsilon])$$

(26) $$\langle \bar{a}'(u), v(u) \rangle \geq 0 \quad \text{on } N_\alpha$$

(27) $$\langle \bar{a}'(u), v(u) \rangle \geq \max(4\varepsilon, \frac{\tau k^2}{4} \|v(u)\|)$$

on $a_*^{-1}([\beta - \varepsilon, \beta + \varepsilon]) - U_{\delta/2}$

(28) v is odd if a, b are even.

Proof. By Lemma 5 there exist locally Lipschitz continuous functionals $g, \bar{g} : X \to [0,1]$ such that

$$g(u) = \begin{cases} 1 & \text{if } u \in a_*^{-1}([\beta - \varepsilon, \beta + \varepsilon]) \\ 0 & \text{if } u \notin a_*^{-1}(]\beta - 2\varepsilon, \beta + 2\varepsilon[) \end{cases}$$

$$\bar{g}(u) = \begin{cases} 1 & \text{if } u \in N_\alpha - U_{\delta/4} \\ 0 & \text{if } u \in U_{\delta/8} \end{cases}$$

$g, \bar{g}$ are even if a, b are even.

Furthermore, define the local Lipschitz continuous function

$$h(s) = \begin{cases} 1 & \text{if } s \in [0,1] \\ s^{-1} & \text{if } s > 1 \end{cases} .$$

Then

$$v(u) = g(u)\bar{g}(u)h(\|p(u)\|)p(u)$$

has all properties claimed in Lemma 8.

We check (27). Suppose

$$u \in a_*^{-1}([\beta - \varepsilon, \beta + \varepsilon]) - U_{\delta/2} .$$

Then $g(u) = \bar{g}(u) = 1$ and, by Lemma 7, 3, 6,

$$\langle \bar{a}'(u), v(u) \rangle \geq h(\|p(u)\|) \frac{k^2}{2} \|\bar{a}'(u)\|^2$$

$$\geq h(\|p(u)\|) \frac{k^2}{4} \|\bar{a}'(u)\|_* \|p(u)\|$$

$$\geq \frac{\tau k^2}{4} \|v(u)\| .$$

If $\|p(u)\| \leq 1$, then $h(\|p(u)\|) = 1$, and by Lemma 7, 3, 6,

$$\langle \bar{a}'(u), v(u) \rangle \geq \frac{k^2}{2} \|\bar{a}'(u)\|^2$$

$$\geq \frac{k^2}{2} \|\bar{a}'(u)\|_*^2 \geq \frac{\tau^2 k^2}{2} \geq 4\varepsilon .$$

If $\|p(u)\| > 1$, then $h(\|p(u)\|) = \|p(u)\|^{-1}$, and by Lemma 7, 3, 6,

$$\langle \bar{a}'(u), v(u) \rangle \geq \frac{k^2}{2} \|\bar{a}'(u)\|^2 / \|p(u)\|$$

$$\geq \frac{k^2}{4} \|\bar{a}'(u)\| \geq \frac{k^2}{8} \|p(u)\| \geq \frac{k^2}{8} \geq 4\varepsilon .$$

q.e.d.

Now construct the deformation $d(u,t)$ by solving the ordinary differential equation

(29) $$\frac{\partial d(u,t)}{\partial t} = v(d(u,t)), \quad d(u,0) = u .$$

Lemma 9. $d(u,t)$ is defined for all $u \in N_\alpha$, $t \in \mathbb{R}$ and belongs to N_α. d is continuous on $N_\alpha \times \mathbb{R}$.

Proof. v is local Lipschitz continuous. Therefore, (29) has local solutions. Since v is bounded on N_α, the local solutions can be continued (see Dieudonné [13] 10.5.5). $d(u,t)$ belongs to N_α since

$$\frac{\partial b(d(u,t))}{\partial t} = \langle b'(d(u,t)), v(d(u,t)) \rangle = 0$$

$$b(d(u,0)) = \alpha \quad \text{(see (24))} .$$

The solutions of (28) are continuously dependent on the initial values (see [13] 10.8.1). From this it follows the continuity of d.

q.e.d.

Proof of (10). Observe (25).

Proof of (11). This follows from Lemma 9 and the semigroup property of solutions of (29).

<u>Proof of (12)</u>. We set $\varphi(t) = a(d(u,t))$, $t \geq 0$. Then, by (9), (24), (26),

$$\varphi'(t) = (a'(d(u,t)), v(d(u,t)))$$

$$= (\bar{a}'(d(u,t)), v(d(u,t))) \geq 0 \ .$$

<u>Proof of (13)</u>. Suppose $\varphi(0) \geq \beta - \varepsilon$, $u \in N_a - U_\delta$. We have to prove $\varphi(1) \geq \beta + \varepsilon$.

Since $\varphi(1) \geq \varphi(0)$ by (12) it suffices to consider elements u with $\varphi(0) < \beta + \varepsilon$, i.e. $u \in a_*^{-1}([\beta - \varepsilon, \beta + \varepsilon]) - U_\delta$.

From (25), (29) it follows $|\varphi(t) - \beta| \leq 2\varepsilon$. Hence $\varphi(t) - \varphi(0) \leq 3\varepsilon$ for all $t \geq 0$.

Set $V = a_*^{-1}([\beta - \varepsilon, \beta + \varepsilon]) - U_{\delta/2}$. Consider an arbitrary real number $s > 0$ such that $d(u,t) \in V$ for all $t \in [0,s]$. For example, all small $s > 0$ have this property. Then we have, by (27), (29),

(30) $$3\varepsilon \geq \varphi(s) - \varphi(0) = \int_0^s \varphi'(t)dt$$

$$= \int_0^s (\bar{a}'(d(u,t)), v(d(u,t)))dt$$

$$\geq \frac{\tau k^2}{4} \left\|\int_0^s v(d(u,t))dt\right\| = \frac{\tau k^2}{4} \|d(u,s) - u\| \ .$$

(31) $$3\varepsilon \geq \varphi(s) - \varphi(0) \geq 4s\varepsilon \quad \text{(see (27))} \ .$$

(30) and (18) show that

$$\|d(u,s) - u\| < \frac{\delta}{8}, \text{ i.e. } t \to d(u,t)$$

cannot enter $U_{\delta/2}$.

(31) shows that $t \to d(u,t)$ leaves V before $t = 1$. Therefore $\varphi(1) \geq \beta + \varepsilon$ by (12).

<u>Proof of (14)</u>. If a, b are even, then v is odd. Hence d is odd in u.

The proof of Proposition 1 is complete.

Corollary 2 follows in the same way. Replace v by $-v$ in (29).

3. A critical point principle.

Consider

$$\beta = \sup_{K\in\Lambda} \inf_{u\in K} a(u)$$
$$\rho = \inf_{K\in\Lambda} \sup_{u\in K} a(u) .$$

Proposition 2. Suppose:

(i) Assumption I_α is satisfied.

(ii) Λ is a nonempty class of subsets of N_α which is invariant under the deformation d constructed in Proposition 1 (resp. Corollary 2), i.e. $K \in \Lambda$ implies $d(K,1) \in \Lambda$.

(iii) $\beta \neq \pm\infty$, (PS_β) holds (resp. $\rho \neq \pm\infty$, (PS_ρ) holds).

Then: $a(\cdot)$ has a critical point u on N_α with $a(u) = \beta$ (resp. $a(u) = \rho$).

Proof. Suppose there is no critical point with $a(u) = \beta$. Then $E_\beta = \phi$. Choose $K \in \Lambda$ with $\inf_{u\in K} a(u) \geq \beta - \varepsilon$. Then, $\inf_{u\in d(K,1)} a(u) \geq \beta + \varepsilon$ by Proposition 1 with $U = \phi$. Since $d(K,1) \in \Lambda$, $\inf_{u\in d(K,1)} a(u) \leq \beta$. This is a contradiction.

The similar proof for ρ employs Corollary 2.

q.e.d.

This critical point principle is more or less used in all papers about Ljusternik-Schnirelman theory. Proposition 2 is a slight modification of the formulation given by Rabinowitz [28].

4. The index.

In order to construct suitable classes Λ we need a topological index.

Let Ψ be the class of all closed symmetric sets not containing the origin in a real Banach space X.

Suppose that to every $M \in \Psi$ there belongs an integer $0 \leq m \leq \infty$ called ind M such that for all $M, M_1, M_2 \in \Psi$:

(I_1) ind $\phi = 0$; ind $M \geq 1$ if $M \neq \phi$.

(I_2) If M is a nonempty finite set, then ind $M = 1$.

(I_3) ind $(M_1 \cup M_2) \leq$ ind M_1 + ind M_2.

(I_4) If there exists an odd continuous mapping $f = M_1 \to M_2$, in particular if $M_1 \subset M_2$, then ind $M_1 \leq$ ind M_2.

(I_5) If M is compact, then ind $M < \infty$, and M has an open symmetric neighborhood U with $\overline{U} \in \Psi$ and ind $\overline{U}$ = ind M.

(I_6) ind $M \leq \dim X$

(I_7) If there is an odd homeomorphism of the unit sphere in $\mathbb{R}^n (n \geq 1)$ onto M, then ind $M = n$.

From (I_3) it follows easily.

(I_8) If ind $M_2 < \infty$, then ind $\overline{M_1 - M_2} \geq$ ind M_1 - ind M_2.

Proposition 3. There exists an index with $(I_1) - (I_8)$.

Proof. The notion of genus introduced by Krasnoselskii [20], [21] and reformulated by Coffman [11] has all the claimed properties (see e.g. Rabinowitz [26], Zeidler [34], [36]).

But there is another notion of index introduced by Fadell, Rabinowitz [14], [15] via cohomology which has all the properties noted above as well. The genus of a set is bigger or equal to the index of a set in the sense of Fadell, Rabinowitz.

Corollary 3. Let X be a real Banach. If an index on X satisfies $(I_1) - (I_7)$, then the following holds:

(I_9) Suppose $P : X \to X_1$ is a linear continuous projector onto the finite-dimensional subspace X_1. Let M be a compact symmetric set in $X - \{0\}$. Then, $\operatorname{ind} M > \dim X_1$ implies $\operatorname{ind} M \cap (I - P)(X) \geq \operatorname{ind} M - \dim X_1$.

Proof. Set $N = M \cap (I - P)(X)$. The case $X_1 = \{0\}$ is trivial. Assume $\dim X_1 \geq 1$. By (I_5) there exists a symmetric neighborhood $U \supset N$ with $\operatorname{ind} \overline{U} = \operatorname{ind} N$. By (I_3), $\operatorname{ind} M \leq \operatorname{ind} (M - U) + \operatorname{ind} \overline{U}$. Obviously, $0 \notin P(M - U)$. Hence, by $(I_4)(I_6)$, $\operatorname{ind} M - U \leq \operatorname{ind} P(M - U) \leq \dim X_1$.

q.e.d.

5. Main Theorems.

Suppose Assumption I_α holds. Consider the problem

(P) $\qquad a'(u) - \lambda b'(u) = 0 \ , \quad u \in N_\alpha \ , \quad \lambda \in \mathbb{R}$

where α is a fixed real number.

If (u,λ) satisfies (P) we will say that (u,λ) is an eigensolution and u is an eigenvector. Then $\lambda = (a'(u),u)/(b'(u),u)$.

E_β denotes the set of all eigenvectors of (P) with $a(u) = \beta$.

Lemma 1 shows that E_β is identical with the set of all critical points of $a(\cdot)$ on N_α with $a(u) = \beta$.

Define for $m = 1,2,\ldots$

$$\Lambda_m = \{K \subseteq N_\alpha : K \text{ compact, symmetric, ind } K \geq m\}$$

$$\Lambda_m^\pm = \{K \in \Lambda_m : \pm a(u) > 0 \text{ on } K\}$$

$$\pm\bar\beta_m^\pm = \begin{cases} \sup_{K \in \Lambda_m} \inf_{u \in K} \pm a(u) \\ -\infty \quad \text{if} \quad \Lambda_m = \phi \end{cases}$$

$$\pm\beta_m^\pm = \begin{cases} \sup_{K \in \Lambda_m^\pm} \inf_{u \in K} \pm a(u) \\ 0 \quad \text{if} \quad \Lambda_m^\pm = \phi \end{cases}$$

$$\pm\beta^\pm = \sup_{u \in N_\alpha} \pm a(u) \ .$$

Obviously, $\Lambda_1 \supseteq \Lambda_2 \supseteq \ldots$, $\Lambda_1^\pm \supseteq \Lambda_2^\pm \supseteq \ldots$. Hence $\pm\bar\beta_1^\pm \geq \bar\beta_2^\pm \geq \ldots$, $\pm\beta_1^\pm \geq \pm\beta_2^\pm \ldots \geq 0$.

In the following theorems we take either the sign + or − .

Theorem 1. Suppose:

(i) Assumption I_α holds.

(ii) $\pm a(\cdot)$ is bounded from above on N_α.

(iii) $a(\cdot)$ satisfies (PS) on N_α.

Then:

1) $E_{\beta^\pm} \neq \phi$.

2) Suppose a,b are even and $0 \notin N_\alpha$. Then the following holds:

a) If $\pm\bar{\beta}_m^{\pm} = \pm\bar{\beta}_{m+1}^{\pm} = \pm \ldots = \pm\bar{\beta}_{m+p}^{\pm} > -\infty$, $p \geq 0$, then $\operatorname{ind} E_{\bar{\beta}_m^{\pm}} \geq p+1$.

b) If $\pm\bar{\beta}_m^{\pm} > -\infty$, then (P) has at least m distinct pairs of eigenvectors $(u,-u)$.

<u>Theorem 2.</u> Suppose

(i) Assumption I_α holds.

(ii) $\pm a(\cdot)$ satisfies $(PS^{\pm})$ on N_α.

Then:

1) If $\infty > \pm\beta^{\pm} > 0$, then $E_{\beta^{\pm}} \neq \phi$.

2) Suppose a,b are even and $0 \notin N_\alpha$. Then, it holds:

a) If $\infty > \pm\beta_m^{\pm} = \pm\beta_{m+1}^{\pm} = \ldots = \pm\beta_{m+p}^{\pm} > 0$, $p \geq 0$, then $\operatorname{ind} E_{\beta_m^{\pm}} \geq p+1$.

b) If $\infty > \pm\beta_m^{\pm}, \pm\beta_{m+p}^{\pm} > 0$, $p \geq 0$, then (P) has at least p distinct pairs of eigenvectors $(u,-u)$ with $\pm a(u) > 0$.

c) $\pm\beta_m^{\pm} \to \pm 0$ as $m \to \infty$ if the following holds:

X is reflexive and separable, N_α is bounded, $a(\cdot)$ is weakly continuous and $a(0) = 0$.

Define

$$\pm\rho_m^{\pm} = \begin{cases} \inf_{K \in \Lambda_m^{\pm}} \sup_{u \in K} \pm a(u) \\ +\infty \quad \text{if} \quad \Lambda_m^{\pm} = \phi \end{cases}$$

$$\pm\rho^{\pm} = \inf_{u \in N_\alpha} \pm a(u) .$$

Obviously, $0 \leq \pm\rho_1^{\pm} \leq \pm\rho_2^{\pm} \leq \ldots$.

<u>Theorem 3.</u> Suppose:

(i) Assumption I_α holds.

(ii) $\pm a(\cdot)$ satisfies $(PS^{\pm})$ on N_α.

(iii) $\pm a(u) > 0$ on N_α (+ or - on N_α).

Then:

1) If $\pm\rho^{\pm} > 0$, then $E_{\rho^{\pm}} \neq \phi$.

2) Suppose a,b are even and $0 \notin N_\alpha$. Then:

a) If $0 < \pm\rho_m^{\pm} = \rho_{m+1}^{\pm} = \ldots = \rho_{m+p}^{\pm} < \infty$ and $p \geq 0$, then $\operatorname{ind} E_{\rho_m^{\pm}} \geq p+1$.

b) If $0 < \pm\rho_1^{\pm}$ and $\Lambda_m^{\pm} \neq \phi$, then (P) has at least m distinct pairs of eigenvectors $(u,-u)$ with $\pm a(u) > 0$.

The following two Propositions provide additional information about the behaviour of the eigenvalues and eigenvectors.

<u>Proposition 4.</u> Suppose:

(i) Assumption I_α holds.

(ii) $a(u) = \beta$, $u \in N_\alpha$ implies $a'(u) \neq 0$. Then, if (u,λ) is an eigensolution of (P) with $a(u) = \beta$, i.e. $u \in E_\beta$, then $\lambda \neq 0$.

<u>Proposition 5.</u> Suppose:

(i) Assumption I_α holds; X is reflexive.

(ii) $\{(u_m,\lambda_m)\}$ is an infinite sequence of eigensolutions of (P) with $a(u_m) = \beta_m$, i.e. $u_m \in E_{\beta_m}$, and $\beta_m \to 0$ as $m \to \infty$.

(iii) a' is strongly continuous.

Then:

1) $\lambda_m \to 0$ as $m \to \infty$ if $a(u) = 0$ implies $\langle a'(u),u\rangle = 0$.

2) $u_m \rightharpoonup 0$ as $m \to \infty$ if $a(u) = 0$ implies $u = 0$.

6. Proofs of the main theorems.

We shall only consider the case "+" ("−" follows by replacing a by $-a$).

Proof of Theorem 1, 1). Use Proposition 2 with $A = \{\{u\} : u \in M_\alpha\}$ q.e.d.

Proof of Theorem 1, 2a. First let $p = 0$. Then, $E_{\vec{\beta}_n^+} \neq \emptyset$ follows from Proposition 2 with $A = A_n$. If $K \in A_n$, then $d(K,1) \in A_n$ by Proposition 1, (14) and Proposition 3, (I_4).

If a, b are even, then a', b' are odd. Therefore $E_{\vec{\beta}_n^+}$ is symmetric, closed, and $0 \notin E_{\vec{\beta}_n^+}$ since $0 \notin M_\alpha$. Hence $\operatorname{ind} E_{\vec{\beta}_n^+} \geq 1$.

Now let $p \geq 1$. We will argue by contradiction. Suppose $\operatorname{ind} E_{\vec{\beta}_n^+} \leq p$. Since (PS), $E_{\vec{\beta}_n^+}$ is compact. By Proposition 3, (I_5) there exists an open set $U \supset E_{\vec{\beta}_n^+}$ with $\operatorname{ind} \bar{U} = \operatorname{ind} E_{\vec{\beta}_n^+}$. By Proposition 1, there exists $\varepsilon > 0$ such that

$$a(u) \geq \vec{\beta}_n^+ - \varepsilon, \quad u \in M_\alpha - U \Rightarrow a(d(u,1)) \geq \vec{\beta}_n^+ + \varepsilon. \tag{32}$$

Since $\vec{\beta}_n^+ = \vec{\beta}_{n+p}^+$ there exists $K \in A_{n+p}$ with $\inf_{u \in K} a(u) \geq \vec{\beta}_n^+ - \varepsilon$. Then

$$\operatorname{ind}(K - U) \geq \operatorname{ind} K - \operatorname{ind} \bar{U} \geq n + p - p = n.$$

by Proposition 3, (I_8). Hence $K - U \in A_n$. Therefore $d(K - U, 1) \in A_n$. Now $a(d(u,1)) \geq \vec{\beta}_n^+ + \varepsilon$ on $K - U$ by (32). But $a(d(u,1)) \leq \vec{\beta}_n^+$ on $K - U$ by construction of $\vec{\beta}_n^+$. This is a contradiction. q.e.d.

Proof of Theorem 1, 2c). $u \in E_\beta$ implies $-u \in E_\beta$. If $\vec{\beta}_n^+ > -\infty$, then $\vec{\beta}_1^+ \geq \dots \geq \vec{\beta}_n^+ > -\infty$. Hence $E_{\vec{\beta}_i^+} \neq \emptyset$, $i = 1, \dots, n$ by Theorem 1, 2a). If all $\vec{\beta}_i^+$ are different from each other then we get n distinct pairs $(u, -u)$ of eigensolutions. If, for example, $\vec{\beta}_1^+ = \vec{\beta}_2^+$, then $\operatorname{ind} E_{\vec{\beta}_n^+} \geq 2$, i.e. $E_{\vec{\beta}_n^+}$ contains an infinite number of eigenvectors by Proposition 3, (I_2).
q.e.d.

Proof of Theorem 2. 1), 2a), 2b) can be proved in a similar way as in the proof of Theorem 1. Observe that $K \in A_n^*$ implies $d(K,1) \in A_n^*$ by Proposition 1, (12).

2c) follows as in Zeidler [34] p. 112, [36] Theorem 1,2 employing a recent result due to Dancer [12] ensuring the existence of nonlinear projection operators in real reflexive separable Banach spaces. Dancer's result shows that every such Banach space has the usual structure in the sense of Fučík, Nečas [16], [17].
q.e.d.

Proof of Theorem 3. Conclude as in the proof of Theorem 1. Employ Corollary 2 instead of Proposition 1. q.e.d.

Proof of Proposition 4. If $\lambda = 0$, then $a'(u) = 0$. This contradicts (ii). q.e.d.

Proof of Proposition 5, 1). Observe $\lambda_n = \langle a'(u_n), u_n \rangle / \langle b'(u_n), u_n \rangle$. (u_n) is bounded by (7). (λ_n) is bounded by (iii), (5). We are going to show that $\lambda = 0$ is the only accumulation point of (λ_n). Let $\lambda_{n'} \to \lambda$ as $n \to \infty$. Then, without any loss of generality we can assume $u_{n'} \rightharpoonup u$. Hence $a(u_{n'}) \to a(u)$, i.e. $a(u) = 0$. Therefore, $\langle a'(u), u \rangle = 0$ and $\lambda = 0$ since $\langle a'(u_{n'}), u_{n'} \rangle \to \langle a'(u), u \rangle$. q.e.d.

Proof of Proposition 5, 2). Suppose $u_{n'} \rightharpoonup u$ is a weakly convergent subsequence. Hence $a(u_{n'}) \to a(u)$, $a(u) = 0$, therefore, $u = 0$. This shows that the whole sequence (u_n) tends weakly to $u = 0$. q.e.d.

7. An important special case.

Let us consider the eigenvalue problem

(33) $$a'(u) = \lambda(b_1'(u) + b_2'(u)), \quad u \in N_\alpha, \quad \lambda \in \mathbb{R}$$

where $\alpha \neq 0$ is a fixed number and $N_\alpha = \{u \in X : b_1(u) + b_2(u) = \alpha\}$.

Assumption II_α. Let α be a fixed real number.

(34) X is a real reflexive Banach space.

(35) $a, b_1, b_2 \in C^1(X,\mathbb{R})$, $b_1(0) = b_2(0) = b_1'(0) = 0$.

(36) b_1' is bounded and uniformly monotone.

(37) a', b_2' are strongly continuous.

(38) b_1', b_2' are local Lipschitz continuous on N_α.

(39) $\inf_{u \in K} |(b_1'(u) + b_2'(u), u)| > 0$ on bounded subsets K of N_α.

(40) $\{u \in N_\alpha : a(u) < \mu\}$ is bounded for all $\mu > 0$.

Corollary 3. Suppose (34), (39) are satisfied. Then (40) is fulfilled if one of the following two conditions holds:

(40a) $a(u) \to +\infty$ if $\|u\| \to \infty$, $u \in N_\alpha$

(40b) $(b_2'(u), u) \geq \varphi(a(u))$ on N_α where $\varphi : \mathbb{R} \to \mathbb{R}$ is decreasing and $\sup_{u \in N_\alpha} (b_1'(u) + b_2'(u), u) < \infty$.

Proof. Ad (40b). For all u with $a(u) < \mu$, $u \in N_\alpha$, we have

$$\text{const} \geq (b_1'(u) + b_2'(u), u) \geq \psi(\|u\|)\|u\| + \varphi(a(u))$$

$$\geq \psi(\|u\|)\|u\| + \varphi(\mu) \quad \text{(see (1))} .$$

Hence $\{u \in N_\alpha : a(u) < \mu\}$ is bounded.

q.e.d.

The condition (40a) (resp. (40b)) is important for applications to Hammerstein integral equations (resp. nonlinear elliptic equations) in Section 9, 10 (resp. Section 11).

Let us consider the case $\alpha < 0$ and $\alpha > 0$.

Proposition 6a. Suppose:

(i) Assumption II_α holds with $\alpha < 0$.

(ii) $a'(u) \neq 0$ and $a(u) > 0$ for all $u \in X$ with $b_1(u) + b_2(u) < 0$.

Then:

1) Equation (33) has a solution (u,λ) with $\lambda \neq 0$.

2) If a, b_1, b_2 are even, and if there exists a symmetric set on N_α which is homeomorphic to the unit sphere in $\mathbb{R}^m$ by an odd homeomorphism, then (33) has at least m distinct pairs of eigenvectors $(u, -u)$ belonging to eigenvalues $\lambda \neq 0$.

Proposition 6b. Suppose:

(i) Assumption II_α holds with $\alpha > 0$, X is separable.

(ii) $a(u) > 0$ and $a'(u) \neq 0$ for all $u \in X - \{0\}$, $a(0) = 0$.

(iii) There exists a r-dimensional subspace $X_1 (1 \leq r < \infty)$ and a linear continuous projector $P : X \to X_1$ onto X_1 such that $N_\alpha \cap (I - P)(X)$ is bounded.

(iv) For all $k \in \mathbb{N}$, there exists a set K_k on $N_\alpha \cap (I - P)(X)$ which is homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd homeomorphism.

Then: The equation (33) has an infinite number of distinct eigensolutions (u_m, λ_m) with $\lambda_m \neq 0$ and $u_m \rightharpoonup 0$ in X, $\lambda_m \to 0$ as $m \to \infty$.

Proof of Proposition 6a. We are going to apply Theorem 3.

Obviously, Assumption I_α is satisfied with $b = b_1 + b_2$.

First we check (PS^+) using Lemma 2. b' satisfies $(S)_1$. Suppose $u_n \rightharpoonup u$ as $n \to \infty$ and (u_n) on N_α with $a(u) = \beta$, $\beta > 0$. Since b_1 is convex and continuous by (36) (see e.g. Zeidler [34] p. 73, p. 65) we have $b_1(u) \leq \underline{\lim}\, b_1(u_n)$ as $n \to \infty$, $b_2(u) = \lim b_2(u_n)$. Hence $b_1(u) + b_2(u) \leq \alpha < 0$. Therefore, $a'(u) \neq$

Lemma 2 shows (PS^+).

Secondly, we prove $\rho^+ = \inf_{u\in N_\alpha} a(u) > 0$. Suppose $\rho^+ = 0$. Then there exists a subsequence (u_n) on N_α such that $a(u_n) \to 0$. (u_n) is bounded by (40). We choose a weakly convergent subsequence $u_{n'} \rightharpoonup u$. Then, $b_1(u) + b_2(u) < 0$ as in the proof of (PS^+) above. Hence $a(u) > 0$ by (ii). This contradicts $a(u_{n'}) \to a(u) = 0$.

Thus, Proposition 6,1) follows from Theorem 3,1) and Proposition 4.

Proposition 6,2) follows from Theorem 3,2b) and Proposition 4 since $\Lambda_m^+ \neq \phi$ by Proposition 3, (I_7) and $\rho_1^+ = \rho^+$.

q.e.d.

Proof of Proposition 6b. Set $X_2 = (I - P)(X)$. Define $\tilde{\Lambda}_m^+ = \{K \in \Lambda_m^+, K \subseteq X_2\}$ and

$$\tilde{\beta}_m^+ = \sup_{K\in\tilde{\Lambda}_m^+} \inf_{u\in K} a(u)$$

$$\beta_m^+ = \begin{cases} \sup_{K\in\Lambda_m^+} \inf_{u\in K} a(u) \\ 0 \text{ if } \Lambda_m^+ = \phi \end{cases}$$

(see Section 5). By (ii), (iv), (I_7), $\tilde{\Lambda}_m^+ \neq \phi$ for all m. Notice $0 \notin N_\alpha$ by (35). Therefore, $\Lambda_m^+ \neq \phi$ for all m, i.e. $\beta_m^+ > 0$ by (ii).

If we replace X by X_2, then $\tilde{\beta}_m^+ \to 0$ as $m \to \infty$ by Theorem 2,2c). Observe that $N_\alpha \cap X_2$ is bounded by (iii). Hence $\tilde{\beta}_m^+ < \infty$ for all m by (37).

Suppose $K \in \Lambda_m^+$, $m > r$. Corollary 3 shows $K \cap X_2 \in \tilde{\Lambda}_{m-r}^+$. Since

$$\inf_{u\in K} a(u) \leq \inf_{u\in K\cap X_2} a(u) \leq \tilde{\beta}_{m-r}^+$$

we get $0 < \beta_m^+ \leq \tilde{\beta}_{m-r}^+$. Therefore, $0 < \beta_m^+ < \infty$ if $m > r$ and $\beta_m^+ \to 0$ as $m \to \infty$.

Now Proposition 6b follows from Theorem 2,2b), and Proposition 4,5. Observe that (PS^+) follows from (ii) and Lemma 2.

q.e.d.

8. Applications to hyperboloids.

Assumption III. Suppose:

(42) $(X, (.,.))$ is a real Hilbert space.

(43) X_1 is a proper finite-dimensional subspace of X.

(44) $P : X \to X_1$ is a linear continuous orthogonal projection onto X_1.

Define

$$Ju = (I - P)u - Pu \equiv (I - 2P)u \quad ,$$

$$b(u) = 2^{-1}(Ju,u). \quad (I = \text{identity}).$$

The set $H_\alpha = \{u \in X : b(u) = \alpha\}$ is said to be a hyperboloid.

If $u = u_1 + u_2$, $u_1 \in X_1$, $u_2 \in (I - P)(X_1)$, then $Ju = u_2 - u_1$, and u belongs to H_α iff $\|u_2\|^2 - \|u_1\|^2 = 2\alpha$.

We consider

(45) $$a'(u) = \lambda Ju,\ u \in H_\alpha,\ \lambda \in \mathbb{R}$$

for a fixed real number α. Obviously, $b' = J$.

Since $J^2 = I$, (45) is equivalent to

(45') $$Ja'(u) = \lambda u,\ u \in H_\alpha,\ \lambda \in \mathbb{R} \quad .$$

Proposition 7. Let $\alpha \neq 0$ be a fixed number. Suppose:

(i) Assumption III holds, $\dim X < \infty$.

(ii) $a \in C^1(X,\mathbb{R})$.

(iii) $a^{-1}(M) \cap H_\alpha$ is bounded if M is bounded in $\mathbb{R}$,

(iv) a is bounded from above on H_α.

Then:

1) (45) has an eigensolution (u,λ).

2) Suppose a is even. Then, (45) has at least m distinct pairs of eigenvectors $(u,-u)$ where

$m = \dim X_1$ if $\alpha < 0$

$m = \dim X - \dim X_1$ if $\alpha > 0$.

Proof. Apply Theorem 1 and Corollary 1. Observe that $\Lambda_m \neq \emptyset$ by Proposition 3 (I_7) since H_α contains the boundary of a m-dimensional ball.

q.e.d.

Proposition 8. Suppose:

(i) $\alpha < 0$ is a fixed number.

(ii) Assumption III is satisfied.

(iii) $a \in C^1(X,\mathbb{R})$, a' is strongly continuous.

(iv) $a(u) \to +\infty$ if $\|u\| \to \infty$, $u \in H_\alpha$.

(v) $(a'(u),u) > 0$ and $a(u) > 0$ if $(Ju,u) < 0$.

Then:

1) Equation (45) has an eigensolution (u,λ) with $\lambda < 0$.

2) If a is even, then (45) has at least $\dim X_1$ distinct pairs of eigenvectors $(u,-u)$ belonging to eigenvalues $\lambda < 0$.

Proposition 8 is a special case of Proposition 6a and was announced by Krasnosel'skii [21], Chapter VI, 16. Set $b_1' = I$, $b_2' = -2P$.

9. Applications to Hammerstein equations.

Consider

(46a) $$K\varphi'(u) = \lambda u, \quad u \in X, \ \lambda \in \mathbb{R}$$

together with the normalization condition

(46b) $$(w,Ju) = 2\alpha \quad \text{for all} \quad w \in (JK)^{-1}(u)$$

where $\alpha < 0$ is a fixed number. X_1 is the eigenspace belonging to the negative eigenvalues of K. J corresponds to X_1 (see Assumption III).

Proposition 9. Suppose:

(i) Assumption III holds.

(ii) $K : X \to X$ is a selfadjoint completely continuous operator which has exactly m negative eigenvalues, $0 < m < \infty$, and $\lambda = 0$ is not an eigenvalue of K.

(iii) $\varphi \in C^1(X,\mathbb{R})$, $\varphi(0) = \varphi'(0) = 0$.

(iv) $\varphi(u) \geq c\|u\|^2$ on X where $c > 0$ is a constant.

(v) $(\varphi'(u),u) > 0$ if $u \neq 0$.

Then:

1) For every $\alpha < 0$, (46) has an eigensolution (u,λ), $\lambda < 0$.

2) Let φ be even. Then, for every $\alpha < 0$, (46) has at least m distinct pairs of eigenvectors $(u,-u)$ belonging to eigenvalues $\lambda < 0$.

We are going to show that Proposition 9 is a special case of Proposition 8. Our proof follows the ideas of Krasnosel'skii [21] (see 46')).

Proof. Let $\{v_i\}$ be the complete orthonormal system of eigenvectors of K. Then

$$Ku = \sum_i \lambda_i (u,v_i)v_i, \quad \lambda_1 \leq \dots \leq \lambda_m < 0 < \lambda_{m+1} < \dots .$$

Set $X_1 = \lim\{v_1,\dots,v_m\}$. If we construct J as in Assumption III, then

$$JKu = \sum_i |\lambda_i|(u,v_i)v_i .$$

JK has only positive eigenvalues. Therefore, $H = (JK)^{1/2}$ exists. Since $J(JK) = (JK)J$, $HJ = JH$. H is completely continuous because K, JK are completely continuous.

Define $a(v) = \varphi(Hv)$ for all $v \in X$. Then, $a' = H\varphi' H$ is strongly continuous.

We need

<u>Lemma 11</u>. There exists a constant $\bar{c} > 0$ such that

$$a(u) \geq \bar{c}\|u\|^2 \quad \text{for all} \quad u \in H_\alpha \quad \text{and all} \quad \alpha < 0 .$$

<u>Proof</u>. Set $X = X_1 \oplus X_2$. If $u = u_1 + u_2$, $u_i \in X_i$, then $Ju = u_2 - u_1$. H maps X_i into X_i.

Since $\dim X_1 < \infty$, and H is bijective on X_1, we get

$$\|Hu_1\| \geq d\|u_1\| \quad \text{for all} \quad u_1 \in X_1$$

where $d > 0$ is a constant. If $u \in H_\alpha$, $\alpha < 0$, then $2^{-1}(\|u_2\|^2 - \|u_1\|^2) = \alpha$. Hence

$$\|u_1\|^2 = \tfrac{1}{2}(\|u_1\| + \|u_1\|^2) \geq \tfrac{1}{2}\|u_2\|^2 + \tfrac{1}{2}\|u_1\|^2$$

$$= \tfrac{1}{2}\|u_1 + u_2\|^2$$

$$\|H(u_1 + u_2)\|^2 = \|Hu_1\|^2 + \|Hu_2\|^2 \geq \|Hu_1\|^2$$

$$\geq d^2\|u_1\|^2 \geq \frac{d^2}{2}\|u_1 + u_2\|^2 .$$

Thus, for all $u \in H_\alpha$, $\alpha < 0$, we have

$$a(u) = \varphi(Hu) \geq c\|Hu\|^2 \geq \frac{cd^2}{2}\|u\|^2 .$$

q.e.d.

We check (v) in Proposition 8. If $(Ju,u) < 0$, then $u \in H_\alpha$, $\alpha < 0$. Lemma 11 shows $a(u) > 0$. Furthermore, $(a'(u),u) = (H\varphi' Hu,u) = (\varphi'(Hu),Hu) > 0$ by (v) in Proposition 9.

Now Proposition 8 ensures solutions of

$$(46') \qquad H\varphi' Hv = \lambda Jv,\ v \in H_\alpha,\ \lambda < 0 .$$

Set $u = Hv$. Then, $H^2\varphi' Hv = \lambda HJv$,

$$JK\varphi'(u) = \lambda Ju,\ K\varphi'(u) = \lambda u ,$$

i.e. u is solution of (46a).

Now we check (46b). Suppose

$$w \in (JK)^{-1}(u), \quad \text{i.e.} \quad u = JKw = H(Hw) .$$

Hence $v = Hw$, and

$$2\alpha = (Jv,v) = (JHw,Hw) = (JH^2w,w)$$

$$= (Kw,w) = (Ju,w) .$$

The proof of Proposition 9 is complete.

q.e.d.

10. Applications to Hammerstein integral equation.

Set $X = L_2(G)$. Consider the integral equation

(47a) $$\lambda u(x) = \int_G k(x,y) f(y,u(y))dy, \quad u \in X, \ \lambda \in \mathbb{R}$$

together with the normalization condition

(47b) $$\int_G Ju(x)w(x)dx = \alpha \quad \text{for all } w \text{ with } JKw = u$$

where $\alpha < 0$ is a fixed number, and $K : X \to X$ is the linear operator produced by the kernel $k(.,.)$. J is defined as in (46).

Proposition 10. Suppose:

(i) G is an open nonempty subset of $\mathbb{R}^n$, $n \geq 1$.

(ii) $f : \mathbb{R}^n \times G \to \mathbb{R}$ satisfies a Carathéodory condition (i.e. $y \to f(y,u)$ is measurable on G for all $u \in \mathbb{R}$, and $u \to f(y,u)$ is continuous on $\mathbb{R}$ for almost all $y \in G$).

(iii) $|f(y,u)| \leq \text{const} + \text{const}\,|u|$ on $G \times \mathbb{R}$; $f(y,0) = 0$ on G.

(iv) $f(y,u)u \geq cu^2$ on $G \times \mathbb{R}$ where $c > 0$ is a constant.

(v) $\int_{G \times G} k^2(x,y)dxdy < \infty$, $k(x,y) = k(y,x)$ if $x, y \in G$.

(vi) K has exactly m negative eigenvalues $0 < m < \infty$ and $\lambda = 0$ is not an eigenvalue of K.

Then:

1) For every $\alpha < 0$, (47) has an eigensolution (u,λ), $\lambda < 0$.

2) Let f be odd in u. Then, for every $\alpha < 0$, (47) has at least m distinct pairs of eigenvectors $(u, -u)$ belonging to eigenvalues $\lambda < 0$.

Proof. Proposition 10 is a special case of Proposition 9 with

$$\phi(u) = \int_G \int_0^{u(x)} f(x,v)dv \ dx .$$

It is easy to show that

$$\phi'(u)(y) = f(y,u(y))$$

(see e.g. Zeidler [34] p. 69).

q.e.d.

11. Applications to elliptic differential equations.

We set

$$Lu \equiv - \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + c(x)u$$

and consider the nonlinear eigenvalue problem

(48) $$Lu(x) = \mu p(u(x),x) \text{ on } G \qquad (x \in G)$$

$$u = 0 \text{ on } \partial G; \quad 2^{-1}\int_G (Lu)u\, dx = \alpha \;.$$

The corresponding linear eigenvalue problem reads as

(49) $$Lu = \mu u \text{ on } G$$

$$u = 0 \text{ on } \partial G.$$

Assumption IV.

(50) G is a smooth bounded nonempty domain in $\mathbb{R}^n$, $n \geq 2$.

(51) L is uniformly elliptic in G, a_{ij} are continuously differentiable in G with Hölder continuous first derivatives.

(52) c is Hölder continuous in G.

(53) p is locally Hölder continuous on $\overline{G} \times \mathbb{R}$.

(54) $|p(u,x)| \leq \text{const} + \text{const}\, |u|^s$ on $\mathbb{R} \times G$ where $1 < s < (n+2)/(n-2)$ if $n > 2$ and $1 < s < \infty$ if $n = 2$; s is fixed.

(55) p is odd in u.

We set $X = \overset{\circ}{W}{}_2^1(G)$ (Sobolev space of $L_2(G)$ - functions with $L_2(G)$ - first generalized derivatives).

Under the Assumption IV, (49) has an infinite number of classical eigensolutions with eigenvalues

$$-\infty < \overline{\mu}_1 \leq \overline{\mu}_2 \leq \ldots \leq \overline{\mu}_m \leq \ldots < \infty$$

and

$$\overline{\mu}_m \to +\infty \text{ as } m \to \infty \;.$$

Proposition 11. Suppose:

(i) Assumption IV holds.

(ii) $p(u,x) > 0$ for all $u > 0$, $x \in G$.

(iii) There exist constants $R_1, \rho > 0$ such that

(56) $$c(x)u^2 \leq R_1 \int_0^u p(v,x)dv + R_2$$

if $|u| \geq \rho$, $x \in G$.

(iv) $\overline{\mu}_1 \leq \overline{\mu}_2 \leq \ldots \leq \overline{\mu}_r < 0 < \overline{\mu}_{r+1} \leq \ldots$

(i.e. (49) has exactly r negative eigenvalues and $\mu = 0$ is not an eigenvalue of (49)).

Then:

1) For every fixed $\alpha < 0$, (48) has at least r distinct pairs of classical eigenfunctions $(u, -u)$ with corresponding eigenvalues $\mu < 0$.

2) For every fixed $\alpha > 0$, (48) has an infinite number of distinct classical eigensolutions (u_m, μ_m) with $\mu_m > 0$ and $\mu_m \to +\infty$ as $m \to \infty$.

Furthermore, $u_m \to 0$ in $X = \overset{\circ}{W}{}_2^1(G)$ as $m \to \infty$.

Remark. The proof will be a consequence of Proposition 6a, 6b. Using the general results due to Browder [8] concerning the properties of operators induced by quasilinear elliptic differential equations we can apply Proposition 6a, 6b to far more general equations than (48).

Proof. It suffices to consider generalized solutions $u \in X$. Then, regularity results ensure, that every solution $u \in X$ of (48), (49) is a classical solution (see Agmon [1], Rabinowitz [27] p. 745, Gilbarg, Trudinger [18]).

Define, for all $u \in X$,

$$b_1(u) = \frac{1}{2} \int_G \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j}\, dx$$

$$b_2(u) = -\frac{1}{2} \int_G c(x)u^2(x)dx$$

$$a(u) = \int_G \left(\int_0^{u(x)} p(v,x)dv \right) dx \;.$$

Then, the generalized problem belonging to (48) reads as

$$\mu a'(u) = b_1'(u) + b_2'(u), \ u \in N_\alpha, \ \mu \in \mathbb{R}$$

where $N_\alpha = \{u \in X : b_1(u) + b_2(u) = \alpha\}$.

$a(\cdot)$ is strongly continuous (see e.g. Zeidler [33] p. 95).

From (ii) it follows

$$a(u) = 0 \Leftrightarrow u = 0 \Leftrightarrow (a'(u),u) = 0$$

because

$$(a'(u),h) = \int_G \phi(u(x),x)h(x)dx \ .$$

Since $0 \notin N_\alpha$, $a(u) > 0$ on N_α if $\alpha \neq 0$ by (ii).

Set $G_1 = \{x \in G : |u(x)| < \rho\}$, $G_2 = G - G_1$. From (56), (ii) it follows, for all $u \in X$,

$$\begin{aligned} 2(b_2'(u),u) &= -\int_G c(x)u^2dx \\ &\geq -\int_{G_1} c(x)u^2dx - \int_{G_2} c(x)u^2dx \\ &\geq \text{const} - R_1 \int_{G_2}\left(\int_0^{u(x)} p(v,x)dv\right)dx \\ &\geq \text{const} - R_1 a(u) \ . \end{aligned}$$

We can choose eigenvectors $u_i \in X$ such that $Lu_i = \bar{\mu}_i u_i$, and, for all $u \in X$, the series $u = \sum_{i=1}^{\infty} c_i u_i$ converge in X with $c_i = \int_G uu_i dx$, $\int_G u_i u_j dx = \delta_{ij}$ (see e.g. Zeidler [33] p. 100). Hence

(57) $$2(b_1(u) + b_2(u)) = \sum_{i=1}^{\infty} \bar{\mu}_i c_i^2$$

since $2(b_1(u) + b_2(u)) = \int_G uLu \, dx$ for all $u \in C^2(\bar{G})$ with $u = 0$ on ∂G.

Define

$$Pu = \sum_{i=1}^{r} c_i u_i, \ c_i = \int_G uu_i \, dx \ .$$

First suppose $\alpha < 0$. Then $P(X) \cap N_\alpha$ is homeomorphic to the unit sphere in $\mathbb{R}^r$ by (57). Now Proposition 11, 1 follows from Proposition 6a. $(a'(u),u) > 0$ if $u \neq 0$ implies $\mu < 0$.

Secondly, suppose $\alpha > 0$. Set

$$K_k = \text{span}\{u_{r+1}, \ldots, u_{r+k}\} \cap N_\alpha \ .$$

Then, K_k is homeomorphic to the unit sphere in $\mathbb{R}^k$ by (57). Furthermore, $b_1(u) + b_2(u) > 0$ on $(I - P)(X) - \{0\}$ by (57). A theorem of Hestenes (cf. [19] Satz 10.2, 10.4) tells us

$$b_1(u) + b_2(u) \geq d\|u\|_X^2 \quad \text{on} \quad (I - P)(X)$$

where d is a positive constant. Hence $N_\alpha \cap (I - P)(X)$ is bounded. Now Proposition 11, 2) follows from Proposition 6b.

q.e.d.

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Let a,b functionals on a real Banach space X. We consider the nonlinear eigenvalue problem

$$a'(u) = \lambda b'(u),\ \lambda \in \mathbb{R},\ u \in N_\alpha = \{u \in X : b(u) = \alpha\}$$

for fixed α. We allow that a', b' are indefinite and the level set N_α is unbounded. L alpha sub alpha

We get finite and infinite lower bounds for the number of eigenvectors on N_α depending on α. Furthermore, we study the weak convergence of the eigenvectors u against zero and the convergence of the eigenvalues λ against zero. next page

lambda

(continued)

alpha

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ABSTRACT (continued)

Our applications are concerned with eigenvalue problems for nonlinear elliptic partial differential equations where the principal elliptical part is indefinite, and with Hammerstein integral equations where the kernel has eigenvalues of different signs.

The main abstract theorems in Section 5 provide a general formulation of the Ljusternik-Schnirelman theory in Banach spaces in the constrained case.